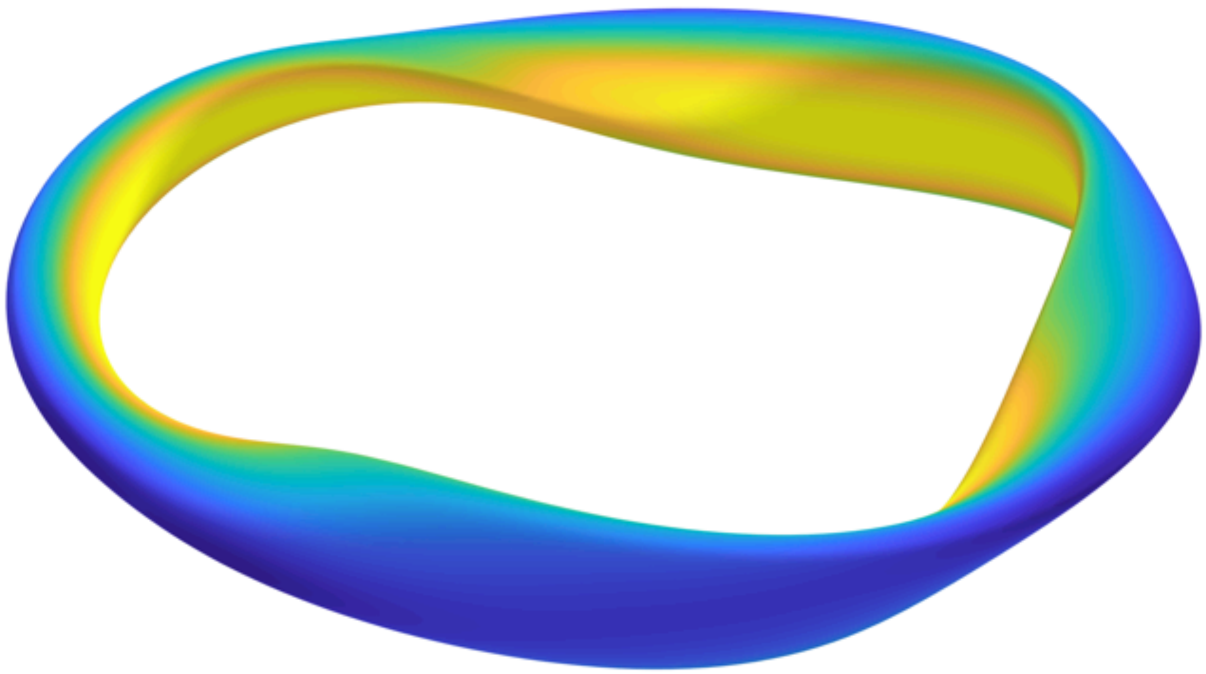
# Optimized stellarators without optimization:

*Direct construction of quasisymmetric shapes*



Matt Landreman (Maryland), Wrick Sengupta (NYU), Gabe Plunk (IPP-Greifswald)

arXiv:1809.10246 https://github.com/landreman/quasisymmetry

- In quasisymmetric designs to date, optimization has been done using “textbook” optimization algorithms to minimize symmetry-breaking Fourier modes in $B$.
  - Many local minima, so result depends on initial guess.
  - Never sure you’ve found all the interesting regions of parameter space.
  - Little insight as to the amount of freedom in the solution.

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  - Little insight as to the amount of freedom in the solution.

- For a complementary approach without these shortcomings, here we extend Garren & Boozer (1991).
  - Usually cited as a proof that quasisymmetry cannot be achieved to $(a/R)^3$.
  - Less well known that it contains a useful constructive procedure.
  - Provides insight & initial conditions for stellopt.

# Main idea of Garren-Boozer: expand position vector using Frenet frame, equate 2 representations of B.

$$\text{Frenet frame } (\mathbf{t},\mathbf{n},\mathbf{b}): \quad \frac{\partial \mathbf{r}_0}{\partial \ell} = \mathbf{t}, \qquad \frac{d\mathbf{t}}{d\ell} = \kappa \mathbf{n}, \qquad \frac{d\mathbf{n}}{d\ell} = -\kappa \mathbf{t} + \tau \mathbf{b}, \qquad \frac{d\mathbf{b}}{d\ell} = -\tau \mathbf{n}$$

$$\mathbf{r}_0 = \text{magnetic axis}, \qquad \kappa = \text{curvature}, \qquad \tau = \text{torsion}$$

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$\mathbf{r}_0=$ magnetic axis, $\quad \kappa=$ curvature, $\quad \tau=$ torsion

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$$\mathbf{r}(r,\theta,\zeta)=\mathbf{r}_0(\zeta)+X(r,\theta,\zeta)\mathbf{n}(\zeta)+Y(r,\theta,\zeta)\mathbf{b}(\zeta)+Z(r,\theta,\zeta)\mathbf{t}(\zeta)$$

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$$\mathbf{B} = B_r \nabla r + B_\theta \nabla\theta + B_\zeta \nabla\zeta, \qquad \mathbf{B} = \nabla\psi \times \nabla\theta + \iota \nabla\zeta \times \nabla\psi$$

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$$\mathbf{B}=B_r\nabla r+B_\theta\nabla\theta+B_\zeta\nabla\zeta, \qquad \mathbf{B}=\nabla\psi\times\nabla\theta+\iota\nabla\zeta\times\nabla\psi$$

Dual relations: $\nabla r=\left[\dfrac{\partial \mathbf{r}}{\partial r}\cdot\dfrac{\partial \mathbf{r}}{\partial \theta}\times\dfrac{\partial \mathbf{r}}{\partial \zeta}\right]^{-1}\dfrac{\partial \mathbf{r}}{\partial \theta}\times\dfrac{\partial \mathbf{r}}{\partial \zeta}$, cyclic permutations.

# Garren & Boozer derived a 1D "Ricatti" ODE for quasisymmetry to O(a/R).

$$\frac{d\sigma}{d\zeta}+\iota\left[\frac{\bar{\eta}^4}{\kappa^4}+1+\sigma^2\right]-2\frac{\bar{\eta}^2}{\kappa^2}\left[I_2-\tau\right]=0$$

$\kappa(\zeta)=$ curvature, $\tau(\zeta)=$ torsion, $I_2=$ current density

$\iota=$ rotational transform, $\bar{\eta}=$ some constant

$$\mathbf{r}(r,\theta,\zeta)=\mathbf{r}_0(\zeta)+rX_{1c}(\zeta)\cos\theta\,\mathbf{n}(\zeta)+r\left[Y_{1s}(\zeta)\sin\theta+Y_{1c}(\zeta)\cos\theta\right]\mathbf{b}(\zeta)+O(r^2)$$

$$X_{1c}(\zeta)=\bar{\eta}/\kappa(\zeta)$$

$$Y_{1s}(\zeta)=\kappa(\zeta)/\bar{\eta}$$

$$Y_{1c}(\zeta)=\sigma(\zeta)\kappa(\zeta)/\bar{\eta}$$

# We have a new proof of existence & uniqueness of solutions, even though the GB ODE is nonlinear.

Given $P(\zeta)>0$, $Q(\zeta)$, and $\sigma(0)$, with $P(\zeta)$ and $Q(\zeta)$

$2\pi$-periodic, bounded, and integrable, a solution to

$$\frac{d\sigma}{d\zeta}+\iota\left(P+\sigma^2\right)+Q=0 \qquad (1)$$

is a pair $\{\iota,\ \sigma(\zeta)\}$ solving $(1)$ where $\sigma(\zeta)$ is $2\pi$-periodic.

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**Theorem:** A solution exists and it is unique.

$\Rightarrow$ Numerical solution is very robust.

# We can now precisely state the amount of freedom in 1st-order-in-r quasisymmetry.

- For every magnetic axis shape (2 functions of $\phi$) with nonvanishing curvature, and 3 real numbers ($\bar{\eta}$, $\sigma(0)$, and $I_2$), there is precisely 1 way to shape the near-axis surfaces consistent with quasisymmetry.
  - For stellarator symmetry, $\sigma(0) = 0$.
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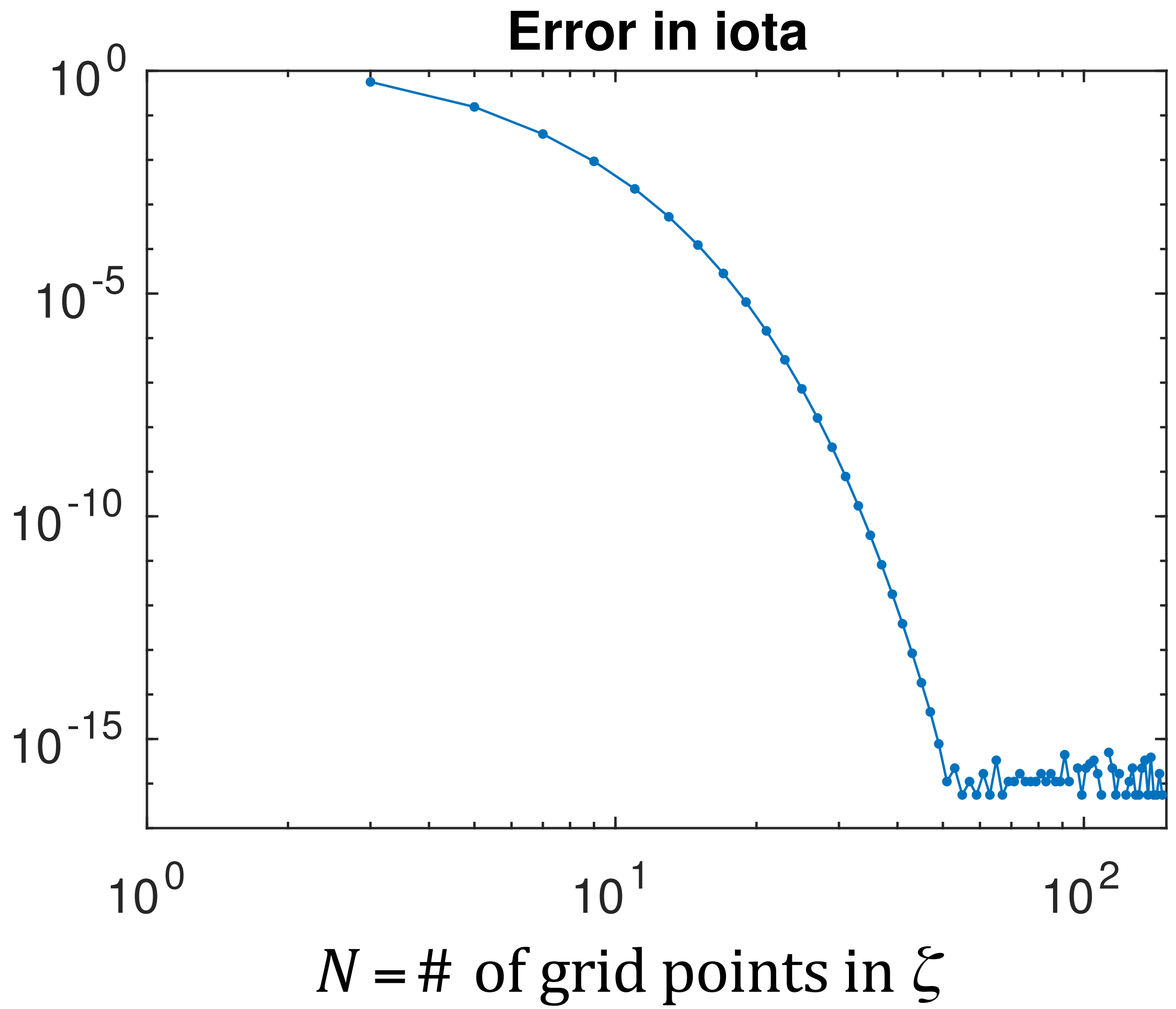
  - In the usual case of no current density on axis, then $I_2 = 0$.
- This solution may be quasi-axisymmetric or quasi-helically symmetric.
- However many of these solutions have absurdly high elongation.

# The ODE is solved with spectral accuracy using pseudospectral discretization + Newton iteration



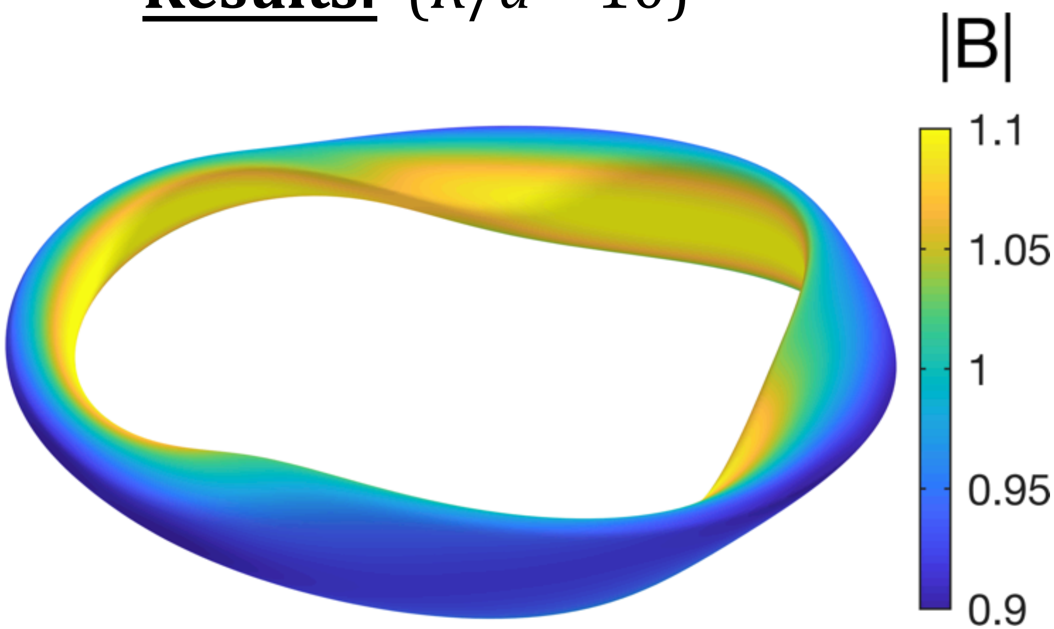
# Quasi-axisymmetric example

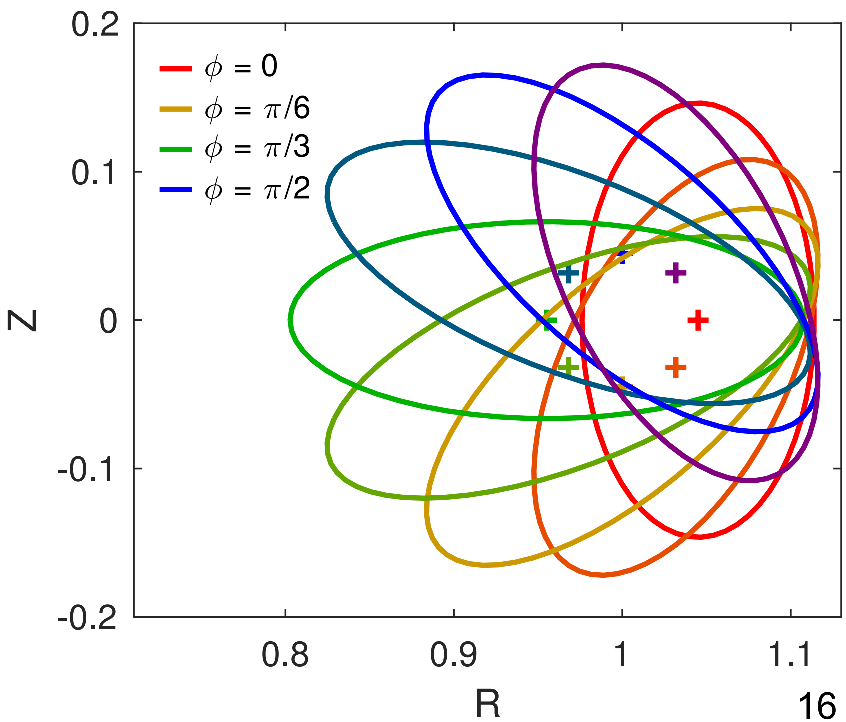
**Inputs:** axis shape $R_0(\phi) = 1 + 0.045\cos(3\phi)$,

$$Z_0(\phi) = -0.045\sin(3\phi),$$

$$\bar{\eta} = -0.9, \qquad \sigma(0) = 0$$

**Results:** $(R/a = 10)$



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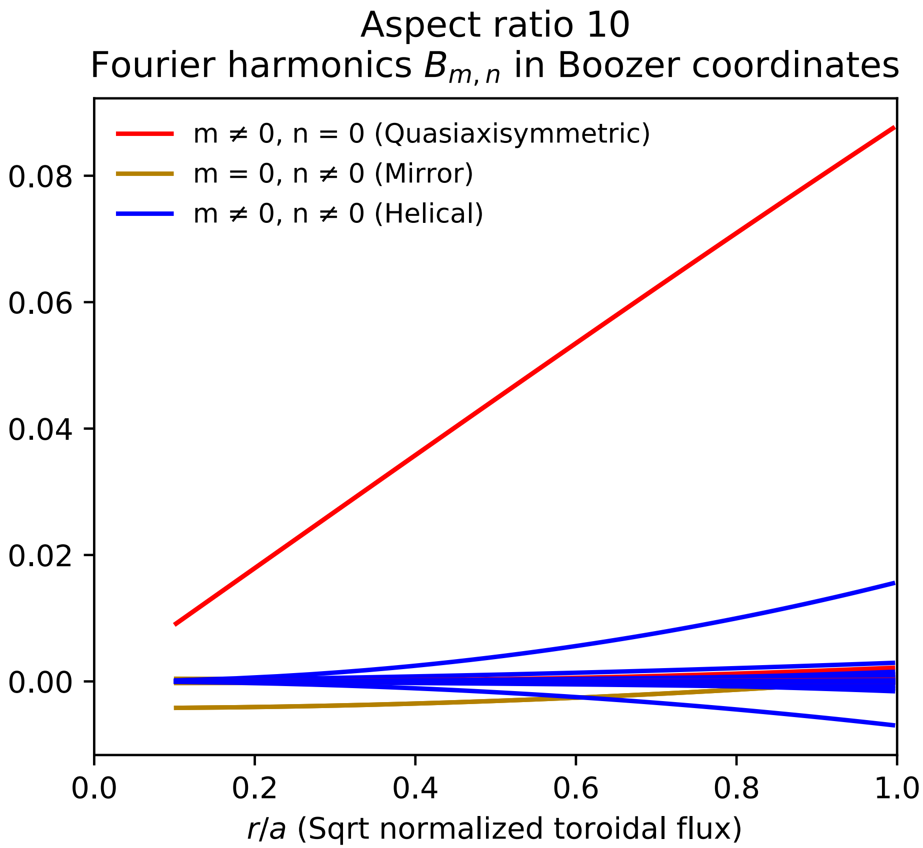
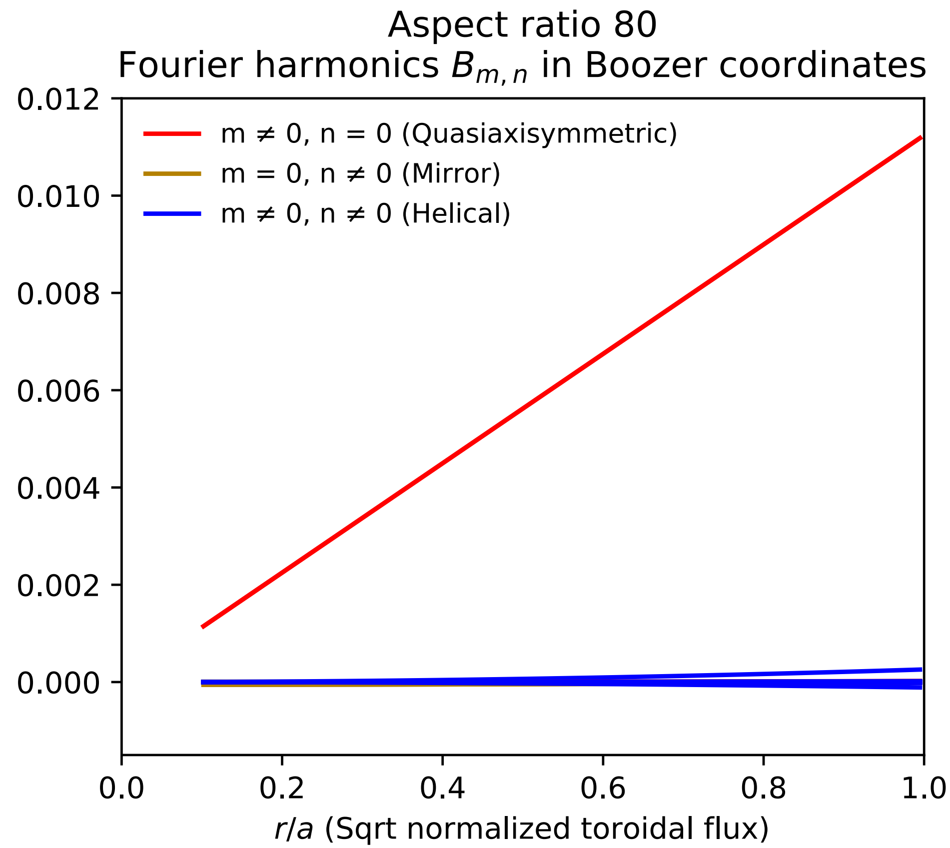
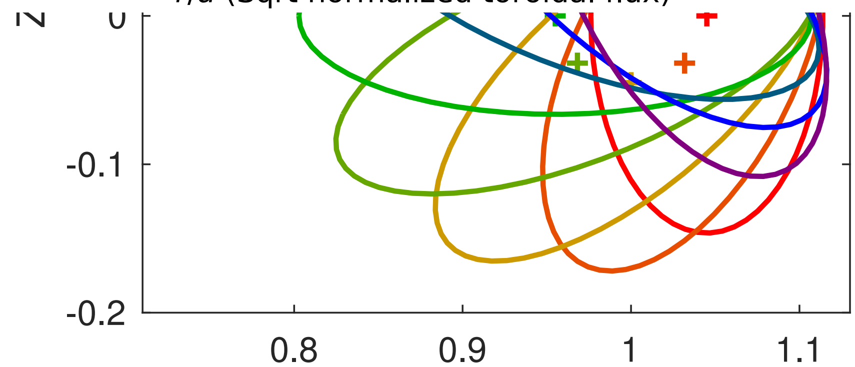
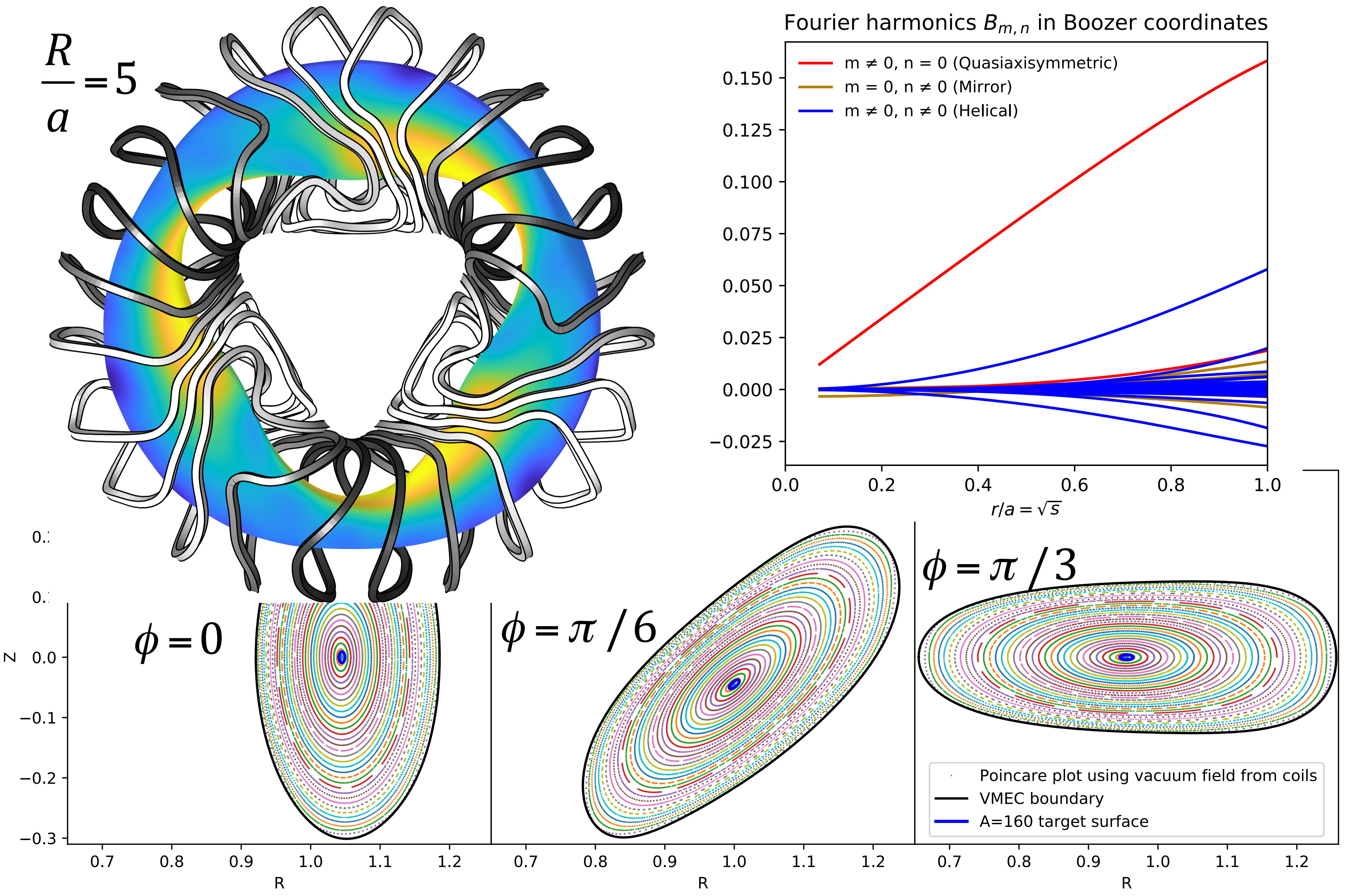






# If you generate coils to make a high-A surface, you actually often get a lower-A stellarator.



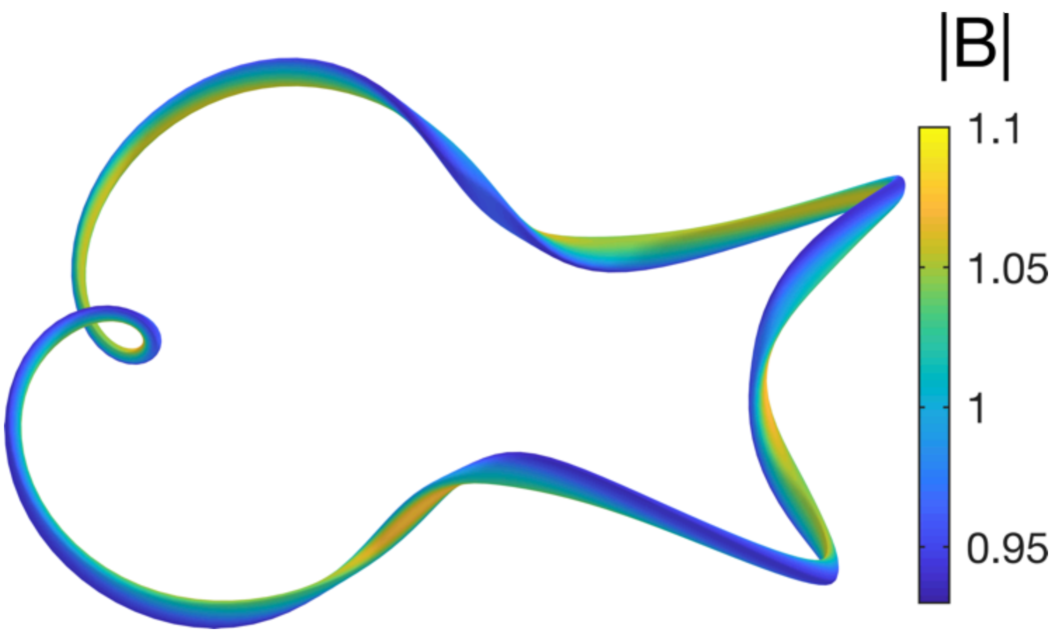
# Quasi-helical symmetry example

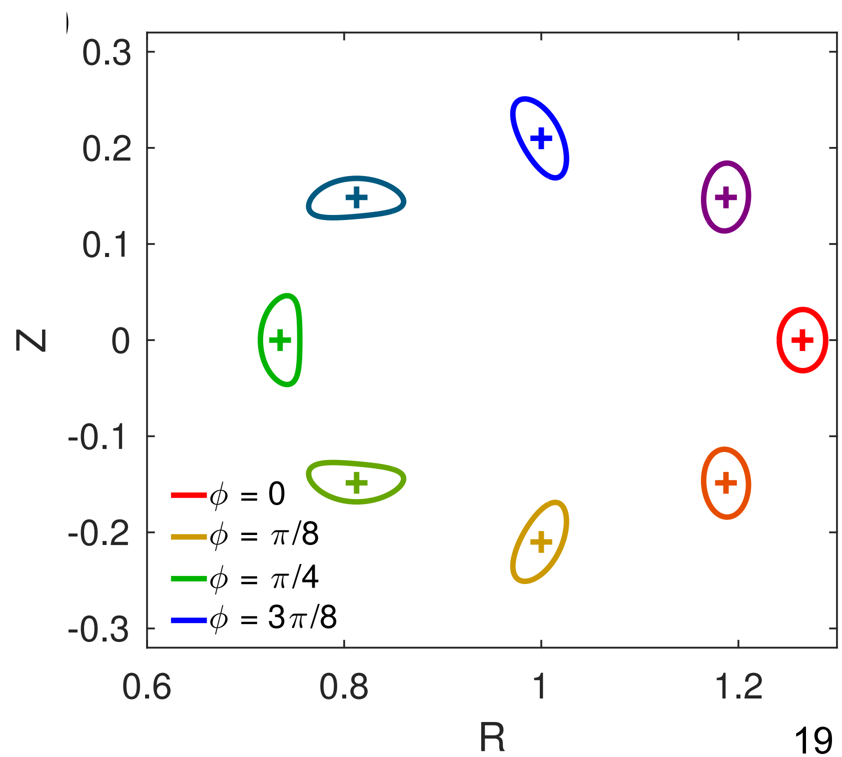
**Inputs:** axis shape $R_0(\phi) = 1 + 0.265\cos(4\phi),$

$$Z_0(\phi) = -0.21\sin(4\phi),$$

$$\bar{\eta} = -2.25, \qquad \sigma(0) = 0$$

**Results:** $(R/a = 40)$



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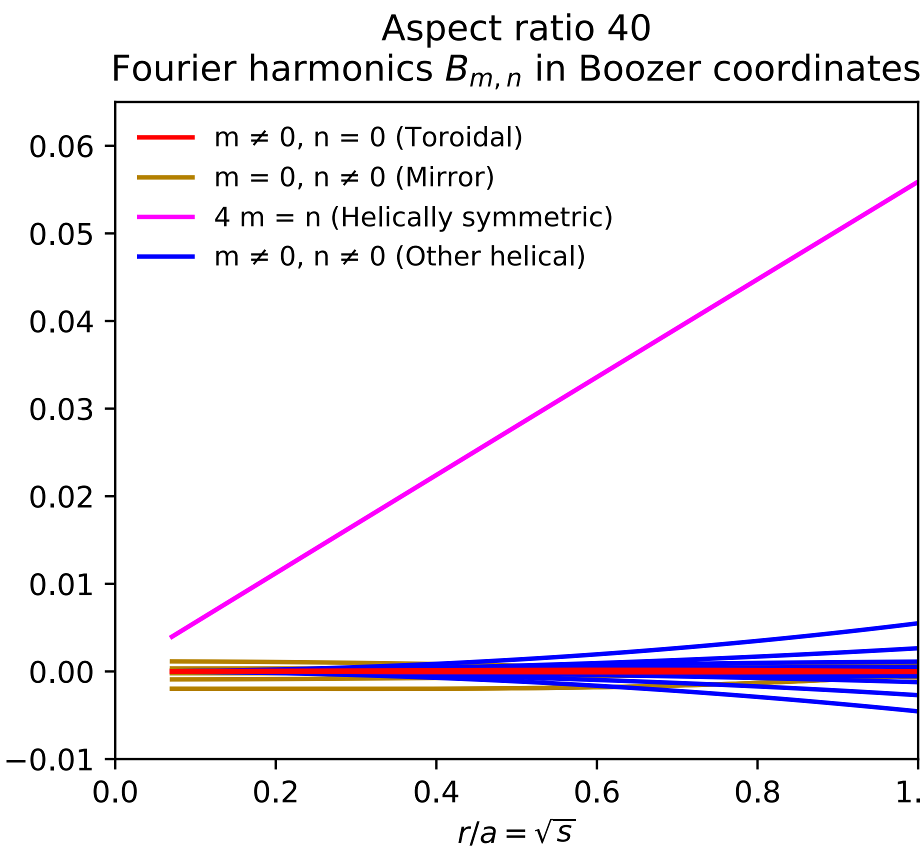
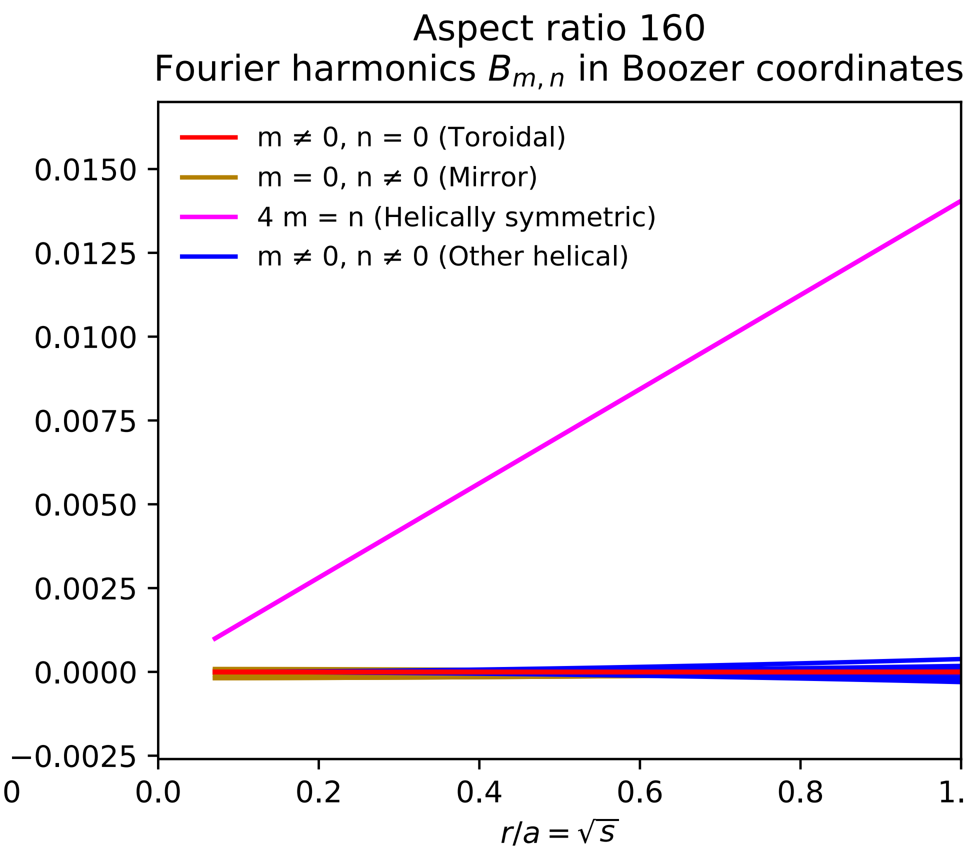
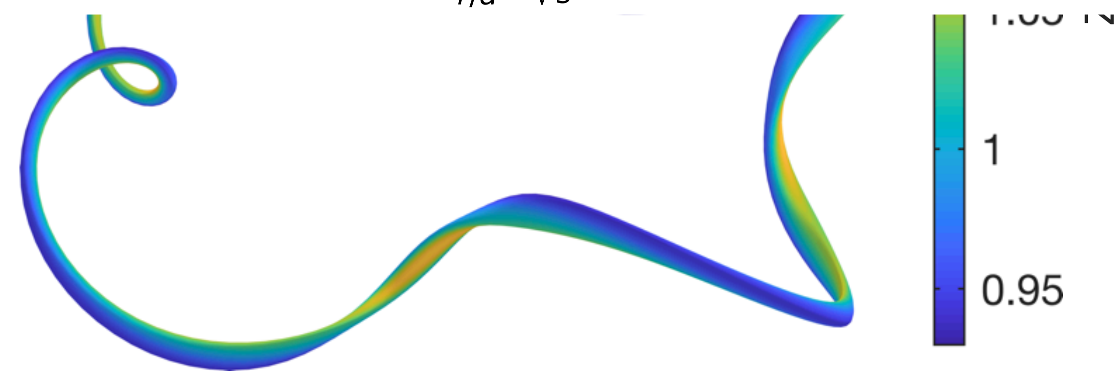
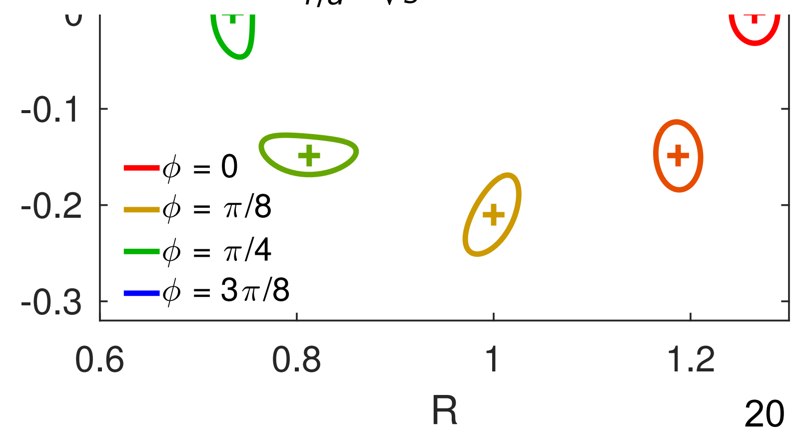


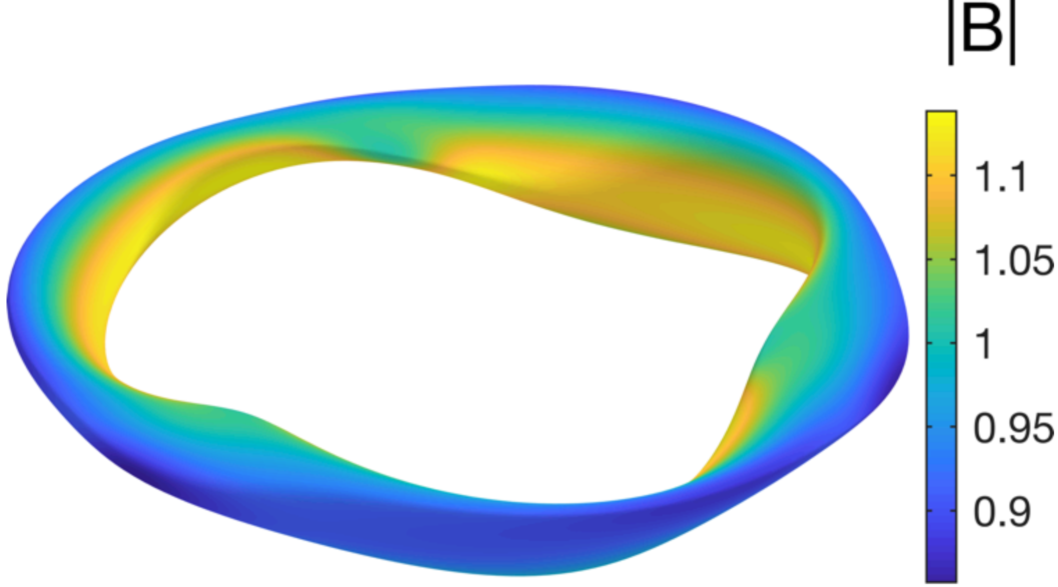
# Idea from Greg Hammett: A quasi-axisymmetric stellarator without stellarator symmetry.

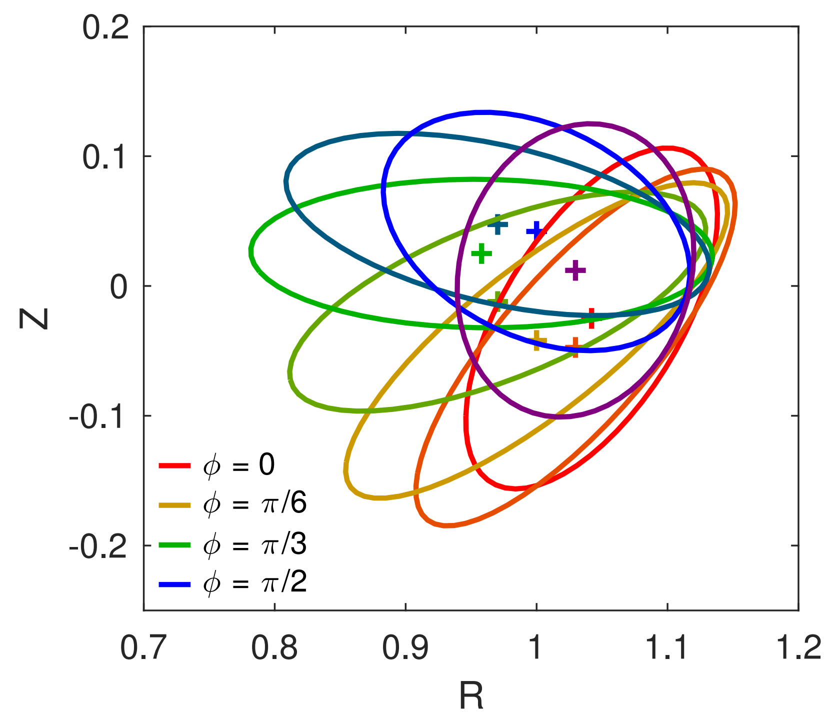
**Inputs:** axis shape $R_0(\phi) = 1 + 0.042\cos(3\phi)$,

$$Z_0(\phi) = -0.042\sin(3\phi) - 0.025\cos(3\phi),$$

$$\bar{\eta} = -1.1, \qquad \sigma(0) = -0.6$$

**Results:** $(R/a = 10)$



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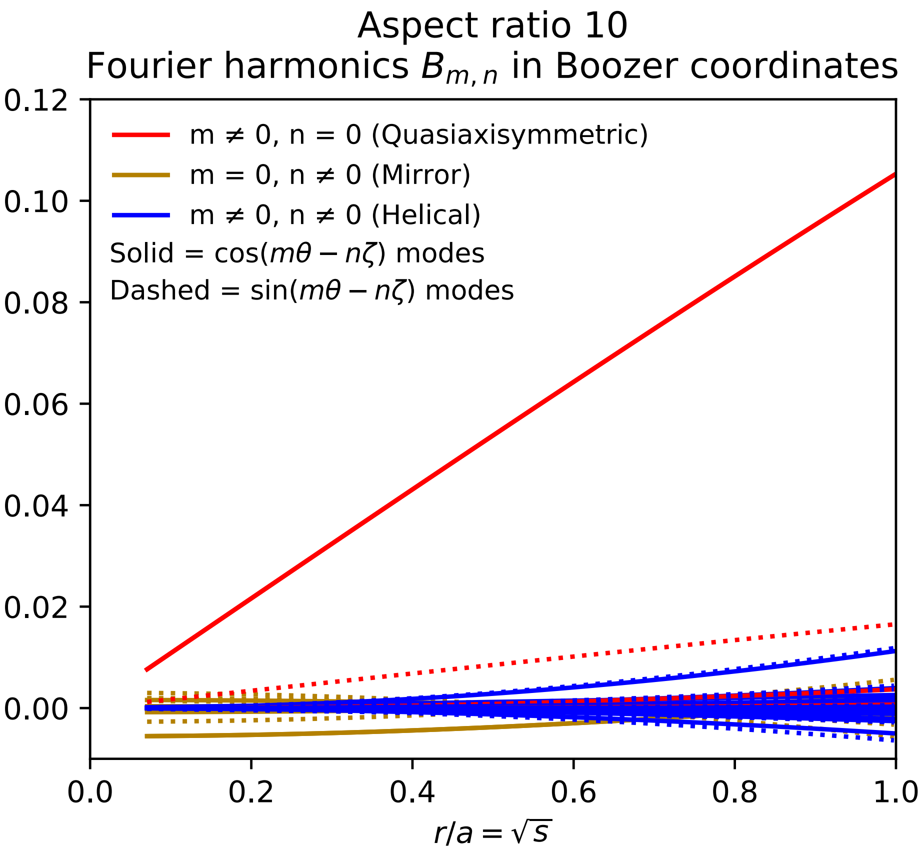
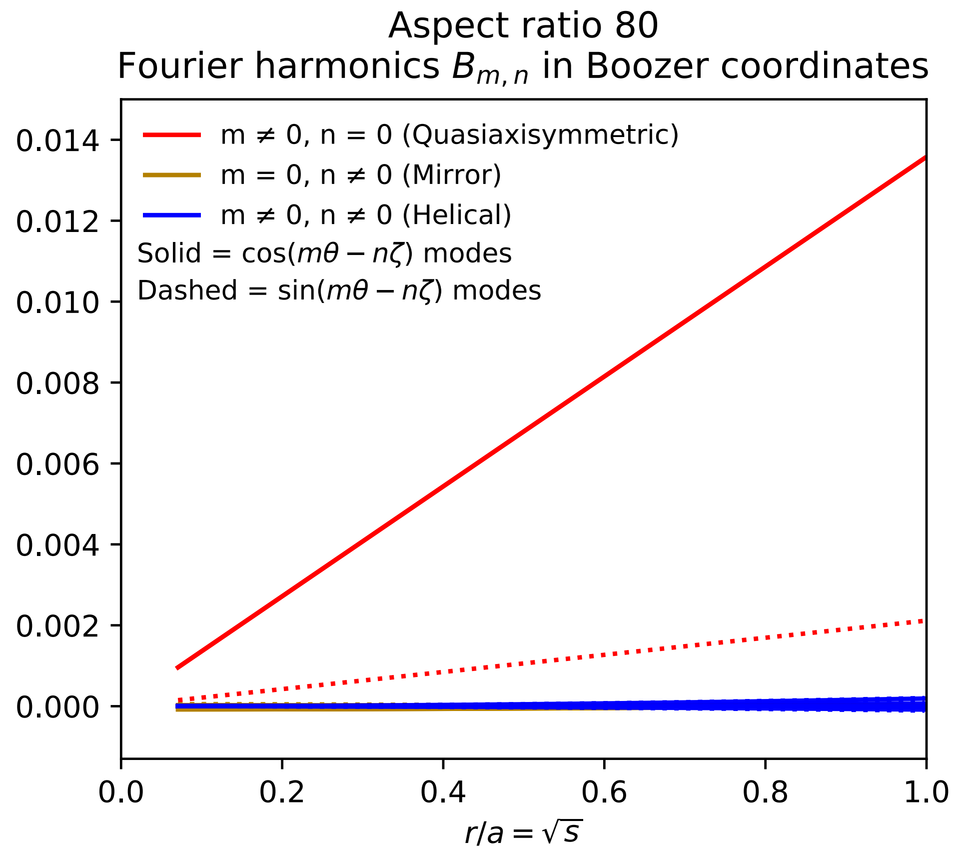
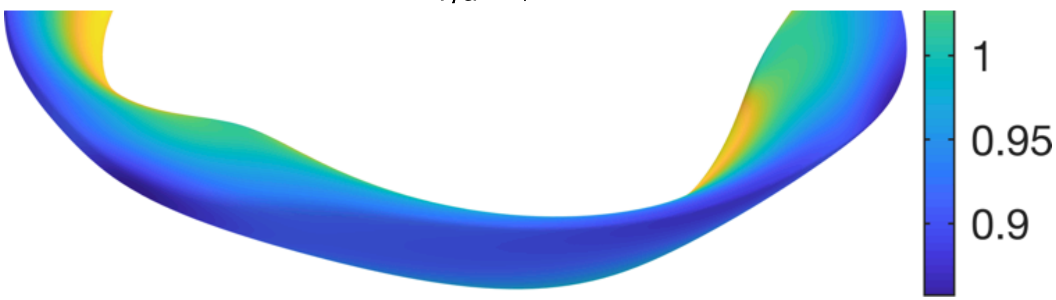
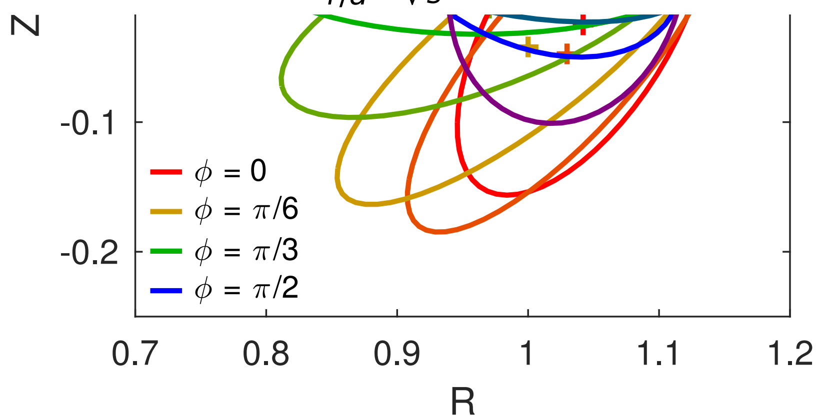
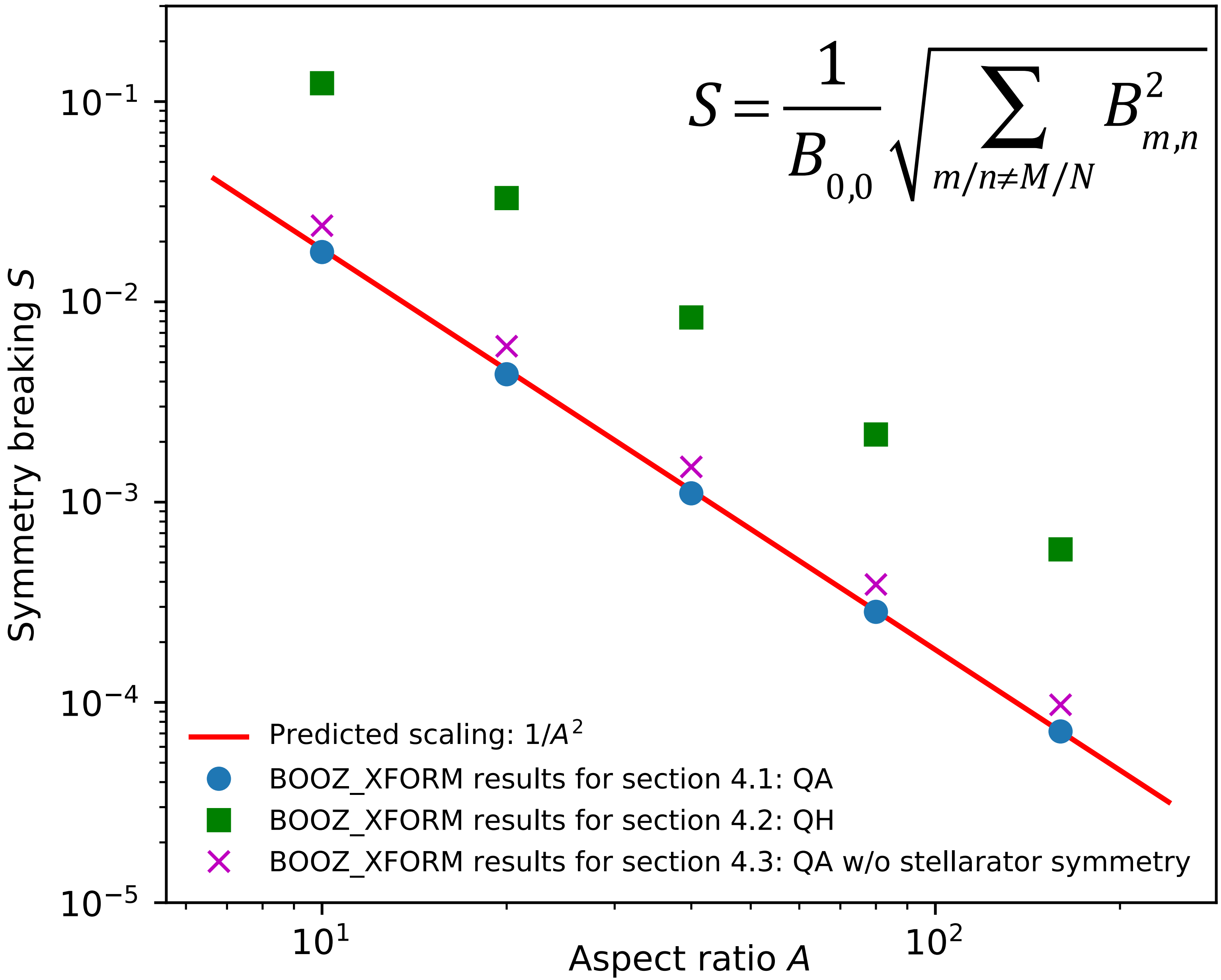


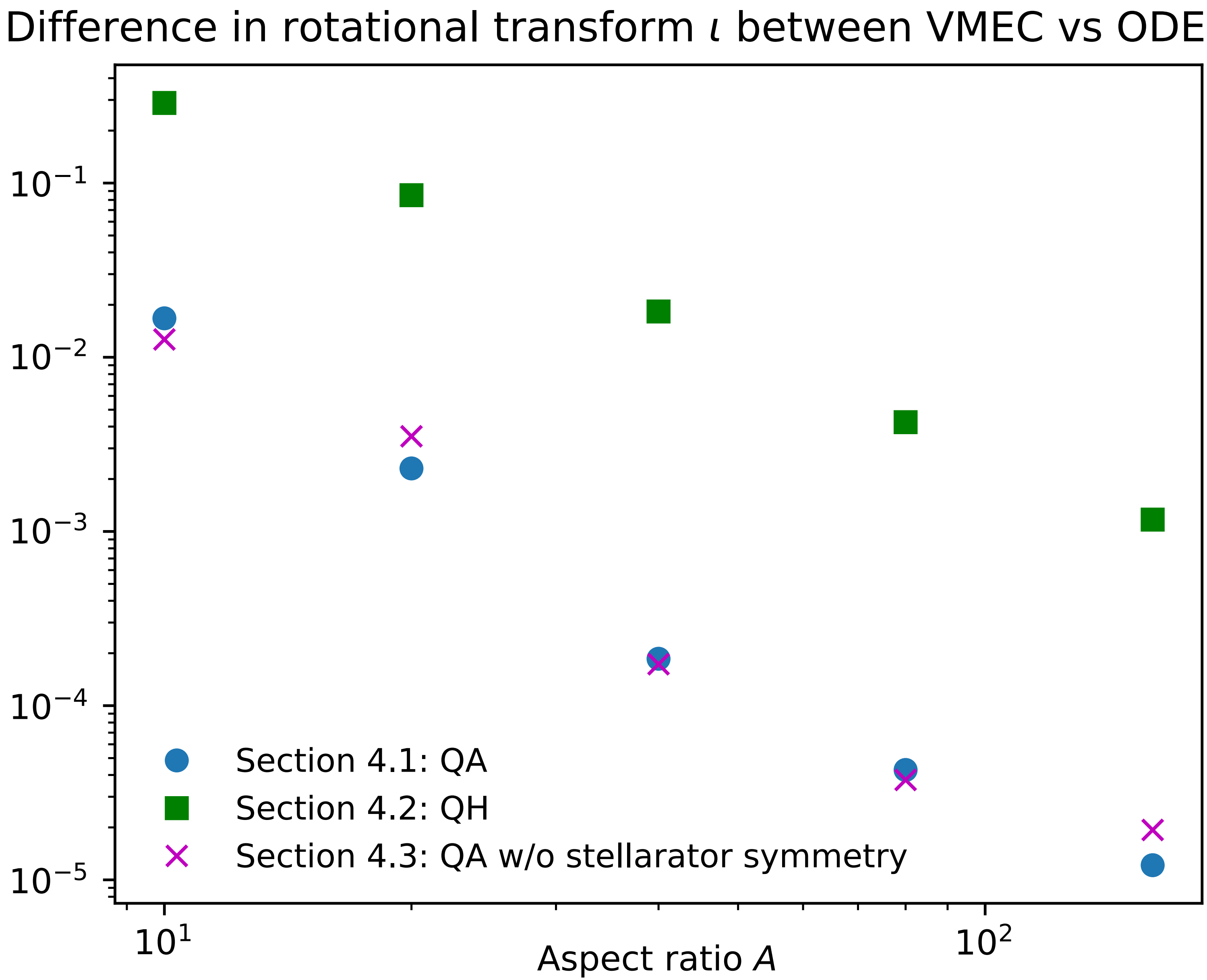



# The symmetry-breaking Fourier amplitudes scale as predicted.



# The rotational transform computed by VMEC converges to the value computed by the Garren-Boozer approach.



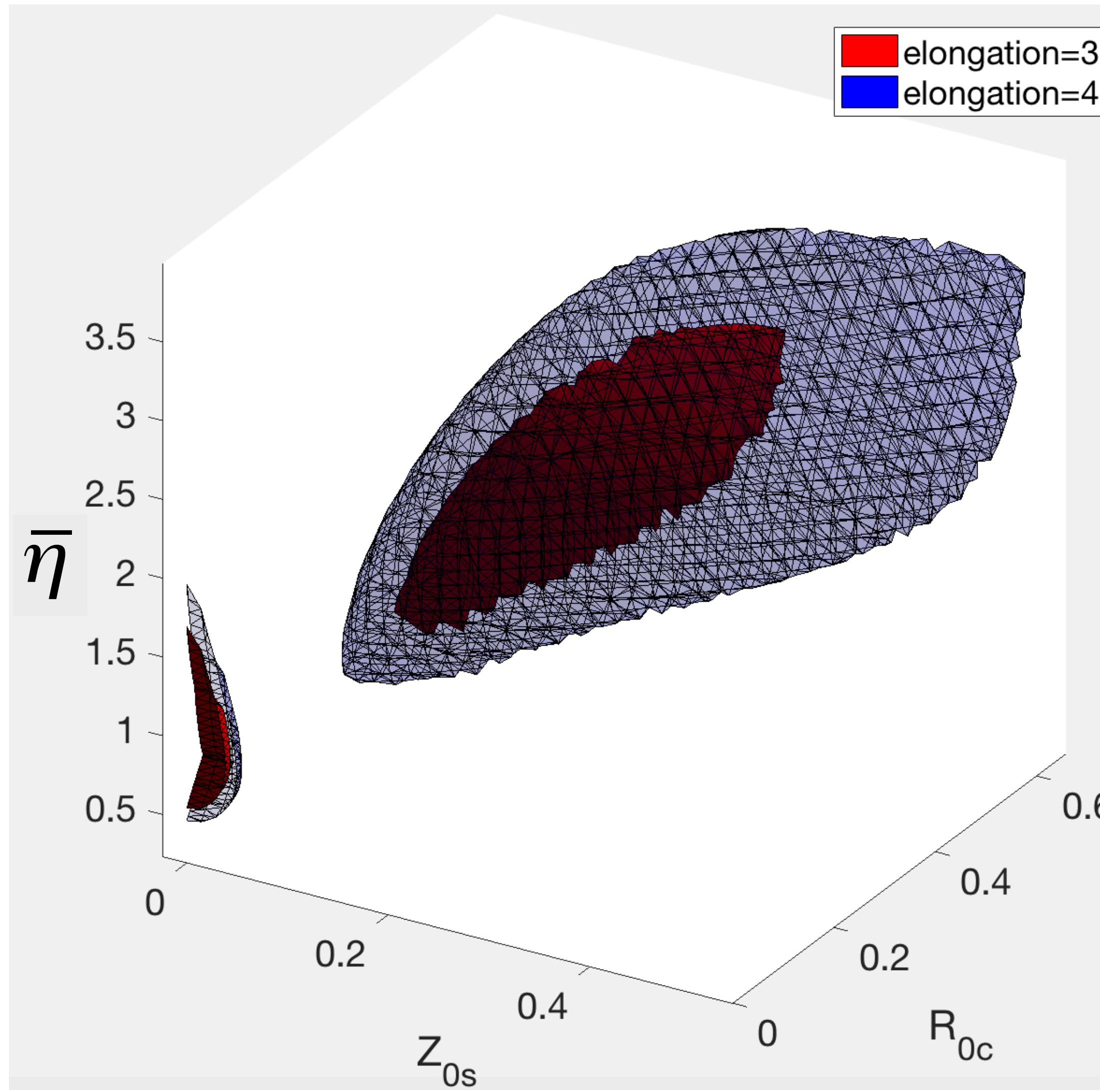
# This method enables fast scans over parameter space.

E.g. Scan over $\left(R_{0c}, Z_{0s}, \bar{\eta}\right)$ where magnetic axis shape is

$$R_0\left(\phi\right) = 1 + R_{0c}\cos\left(4\phi\right)$$

$$Z_0\left(\phi\right) = Z_{1s}\sin\left(4\phi\right)$$

274,560 solutions generated in <30s on my laptop.



*Conclusion:* The Garren-Boozer contstruction provides insight into the space of quasisymmetric shapes, & useful initial conditions for stellopt.

*There are many extensions to pursue:*

- Fully map the landscape of possible 1$^{st}$-order quasisymmetric shapes by considering more Fourier modes in the axis shape. (How do I plot this?)
- Omnigenity.
- Can we construct shapes with quasisymmetry imposed at a mid-radius surface?
- 2$^{nd}$ order in distance from the axis.
- Connect to analysis of the difficulty of producing various plasma shapes.

arXiv:1809.10246 https://github.com/landreman/quasisymmetry

# Extra slides

# The ODE is solved with spectral accuracy using pseudospectral discretization + Newton iteration

Uniform grid in $\phi$ with $N$ points: $\phi_1 = 0,\ \phi_2 = 2\pi / \left(N n_{fp}\right),\ ...,\ \phi_N = 2\pi\left(N-1\right) / \left(N n_{fp}\right)$.

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Vector of $N$ unknowns: $\left(\iota,\ \sigma\left(\phi_2\right),\ \sigma\left(\phi_3\right),\ ...,\ \sigma\left(\phi_N\right)\right)^T$

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$N$ equations: impose ODE at $\phi_1,\ ...,\ \phi_N$.

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$\dfrac{d\sigma}{d\phi} \to D\sigma$ where $D$ is a pseudospectral differentiation matrix.

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