Penalty Functions in FOCUS to Constrain Stellarator Coil Optimization

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HSX

Outline

- Overview of constrained optimization and penalty functions
- FOCUS penalty functions and objective functions
- HSX coils optimized with and without penalty objective functions

We Want to Constrain Important Engineering Quantities in FOCUS

Unconstrained

$$\min \quad f(\boldsymbol{x})$$

Constrained

$$\min \quad f(\boldsymbol{x})$$

$$\text{subject to} \quad g_i(\boldsymbol{x}) \leq 0 \quad \text{for} \quad i = 1, \dots, n$$

$$h_j(\boldsymbol{x}) = 0 \quad \text{for} \quad j = 1, \dots, m$$

We want to constrain the maximum curvature and minimum coil to coil separation in FOCUS

Unconstrained

$$\min \quad w_{Bn} f_{Bn} + w_L f_L$$

→

Constrained

$$\min \quad w_{Bn} f_{Bn} + w_L f_L$$

$$\text{subject to} \quad \kappa \leq \kappa_0 \quad \text{for any point on any coil}$$

$$|\boldsymbol{r_l} - \boldsymbol{r_k}| \geq r_\Delta \quad \text{for any two points on any two coils}$$

Penalty Functions Turn a Constrained Optimization Problem into a Series of Unconstrained Optimization Problems

$$\min \quad f(\boldsymbol{x})$$
$$\text{subject to} \quad g_i(\boldsymbol{x}) \le 0 \quad \text{for} \quad i = 1, \dots, n$$

$$\min \quad \Phi_k(\boldsymbol{x}) = f(\boldsymbol{x}) + \textstyle\sum_{i=1}^{n} p_i(g_i(\boldsymbol{x}), \alpha_{k,i})$$

$$\text{where} \quad p_i(g_i(\boldsymbol{x}), \alpha_{k,i}) > 0 \quad \text{if} \quad g_i(\boldsymbol{x}) > 0$$

$$p_i(g_i(\boldsymbol{x}), \alpha_{k,i}) = 0 \quad \text{if} \quad g_i(\boldsymbol{x}) \le 0$$

$$p_i\big(g_i(\boldsymbol{x}), \alpha_{k,i}\big) \to \infty \quad \text{as} \quad \alpha_{k,i} \to \infty$$

$$p_i\big(g_i(\boldsymbol{x}), \alpha_{k,i}\big) < p_i\big(g_i(\boldsymbol{x}) + |\varepsilon|, \alpha_{k,i}\big) \quad \forall \quad g_i(\boldsymbol{x}) > 0$$

$$0 < \alpha_{k,i} < \alpha_{k+1,i}$$

$$p_i(g_i(\boldsymbol{x}), \alpha_{k+1,i}) > p_i(g_i(\boldsymbol{x}), \alpha_{k,i}) \quad \forall \quad g_i(\boldsymbol{x}) > 0$$

$$\frac{\partial p_i(g_i(x), \alpha_{k,i})}{\partial g_i(x)} \in C^0 \quad \text{at least}$$

$$\min \quad w_{Bn} f_{Bn} + w_L f_L$$
$$\text{subject to} \quad \kappa \le \kappa_0 \quad \text{for any point on any coil}$$
$$|\boldsymbol{r_l} - \boldsymbol{r_k}| \ge r_\Delta \quad \text{for any two points on any two coils}$$

$$\min \quad w_{Bn} f_{Bn} + w_L f_L + p_{cc} + p_\kappa$$

Penalty Functions Used in FOCUS Optimizations

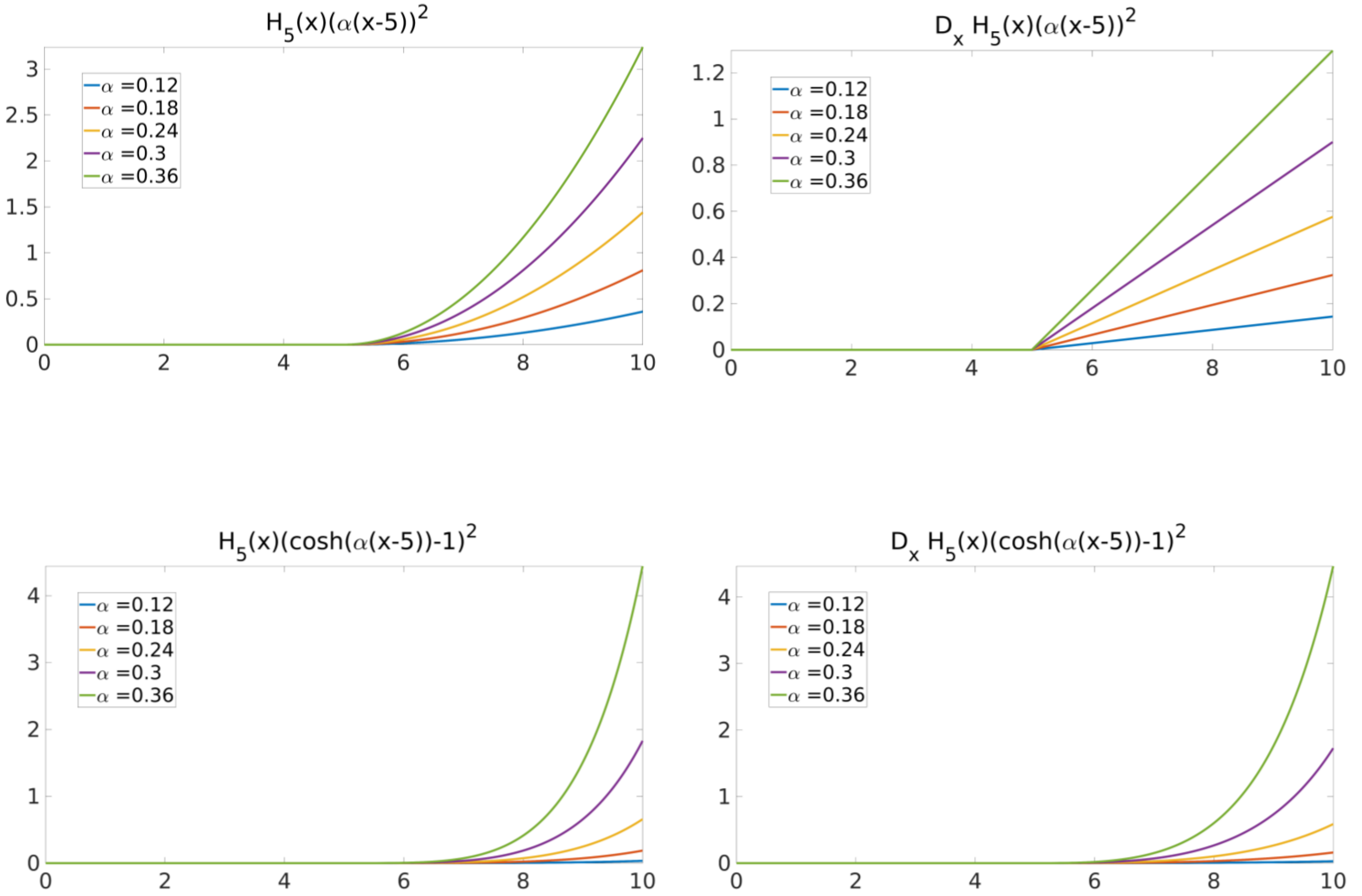
$$p_1(x) = H_{x_0}(x)\big(\alpha(x - x_0)\big)^{\beta}$$

$$\alpha > 0, \qquad \beta \geq 2$$

$$p_2(x) = H_{x_0}(x)\big(\cosh\big(\alpha(x - x_0)\big) - 1\big)^2$$

$$\alpha > 0$$

$$H_{x_0}(x) \equiv H(x - x_0) = \begin{cases} 0, & x < x_0 \\ \frac{1}{2}, & x = x_0 \\ 1, & x > x_0 \end{cases}$$



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FOCUS has Four Curvature Functions Implemented

Linear curvature objective function

$$f_{\kappa,1} = \frac{1}{N_c}\sum_{i=1}^{N_c}\int_0^{2\pi} \kappa_i \, dt$$

Quadratic curvature objective function

$$f_{\kappa,2} = \frac{1}{N_c}\sum_{i=1}^{N_c}\int_0^{2\pi} \kappa_i^2 \, dt$$

Curvature penalty function, hyperbolic cosine penalty function can be substituted with another penalty function

$$p_{\kappa,1} = \frac{1}{N_c}\sum_{i=1}^{N_c}\int_0^{2\pi} H_{\kappa_0}(\kappa_i)(\cosh(\alpha(\kappa_i - \kappa_0)) - 1)^2 \, dt$$

$$\alpha > 0 \qquad \kappa_0 > 0$$

Curvature penalty function and curvature "complexity" objective function. Again, penalty function can be substituted

$$p_{\kappa,2} = \frac{1}{N_c}\sum_{i=1}^{N_c}\frac{1}{L_i}\int_0^{2\pi} |\mathbf{r}_i'| \left(H_{\kappa_0}(\kappa_i)(\cosh(\alpha(\kappa_i - \kappa_0)) - 1)^2 + c\,\kappa_i \right) dt$$

$$\alpha > 0 \qquad \kappa_0 > 0 \qquad c > 0$$

FOCUS can Constrain the Minimum Coil to Coil Separation Using Penalty Functions

$$p_{cc} = \frac{2}{N_c(N_c-1)} \sum_{i=1}^{N_c-1} \sum_{j=i+1}^{N_c} \int_0^{2\pi} \int_0^{2\pi} H_{-r_\Delta}(-|\mathbf{r}_i - \mathbf{r}_j|) \; \left(\cosh\left(\alpha\left(r_\Delta - |\mathbf{r}_i - \mathbf{r}_j|\right)\right) - 1\right)^2 dt_i \, dt_j$$

$$\alpha > 0 \qquad r_\Delta > 0$$

- Penalty function can be substituted with another penalty function
- $\frac{N_c(N_c-1)}{2}$ is equal to the binomial coefficient $\binom{N_c}{2}$, the distinct number of pairs that can be selected from N objects
- Heaviside function steps down

$$H_{-x_0}(-x) \equiv H(x_0 - x) = \begin{cases} 1, & x < x_0 \\ \frac{1}{2}, & x = x_0 \\ 0, & x > x_0 \end{cases}$$

FOCUS Optimizes for a Normal Magnetic Field Distribution on a Plasma Boundary

$$f_{Bn,1} = \int_S \frac{1}{2} \left(\mathbf{B_v} \cdot \mathbf{n} - T_{Bn} \right)^2 dS \qquad f_{Bn,2} = \int_S \frac{1}{2} \left(\frac{\mathbf{B_v} \cdot \mathbf{n} - T_{Bn}}{|\mathbf{B_v}|} \right)^2 dS$$

- $\mathbf{B_v}$ is the magnetic field from coils
- T_{Bn} is a Neumann boundary condition that comes from plasma currents and any fixed external current sources
- S is a plasma boundary
- Coil currents held fixed during optimization and therefore I do not include a toroidal flux objective function

Three Length Objective Functions are Implemented in FOCUS

$$f_{L,1} = \frac{1}{N_c} \sum_{i=1}^{N_c} \frac{1}{2} \frac{(L_i - L_{i,0})^2}{L_{i,0}^2}$$

$$f_{L,2} = \frac{1}{N_c} \sum_{i=1}^{N_c} \frac{e^{L_i}}{e^{L_{i,0}}}$$

$$f_{L,3} = \frac{1}{N_c} \sum_{i=1}^{N_c} \frac{H_{L_\Delta}(\,|L_i - L_{i,0}|)\,(|L_i - L_{i,0}| - L_\Delta)^2}{2\,L_{i,0}^2}$$

$$L_\Delta \geq 0 \qquad L_{i,0} > 0$$

- $L_{i,o}$ is the user-specified ideal coil length
- $f_{L,1}$ and $f_{L,3}$ can be thought of as weak constraints

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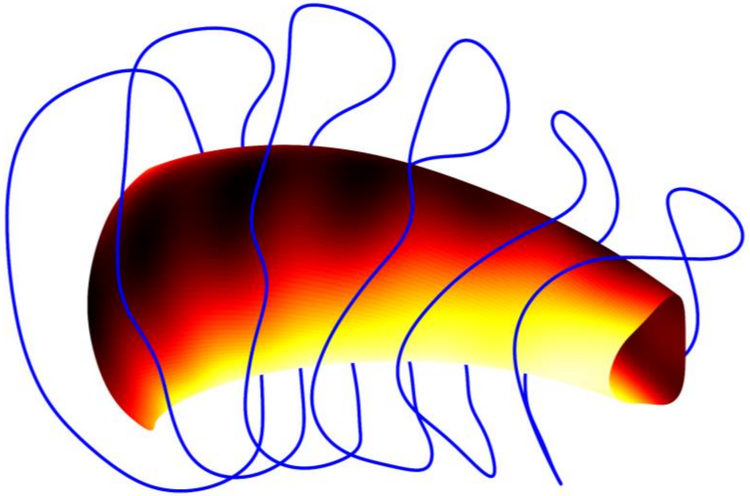
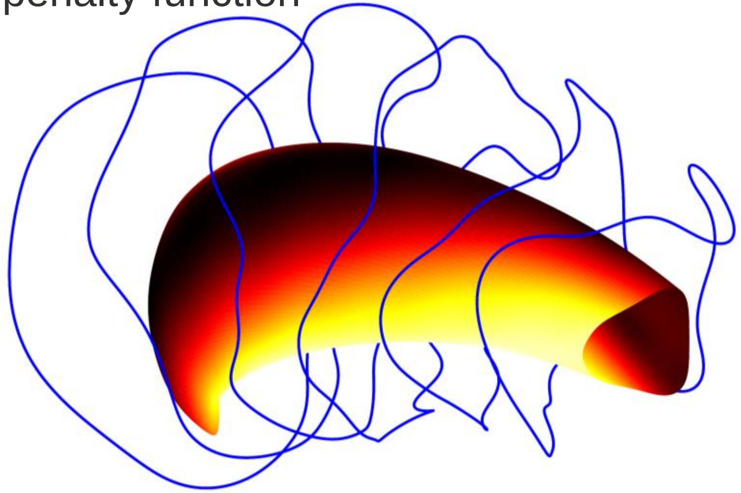
Six HSX Coils Sets are Compared

1. The HSX coils that are not optimized with FOCUS
2. Coils optimized with no curvature functions and no coil to coil separation penalty function
3. Coils optimized with the linear curvature objective function and no coil to coil separation penalty function
4. Coils optimized with the quadratic curvature objective function and no coil to coil separation penalty function
5. Coils optimized with a curvature penalty function and no coil to coil separation penalty function
6. Coils optimized with a curvature penalty function and the coil to coil separation penalty function

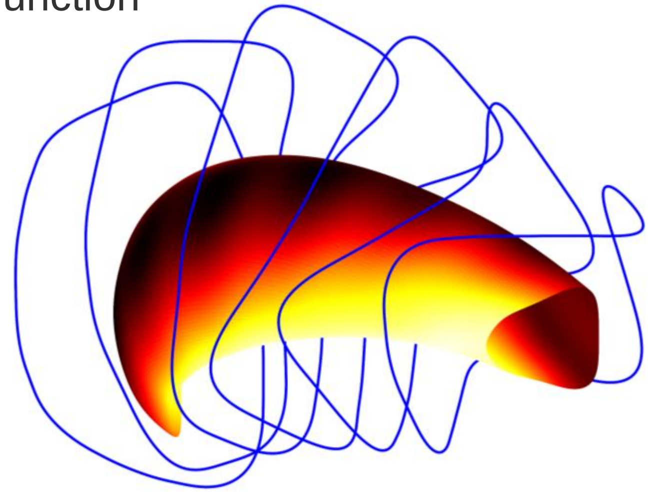
1. The HSX coils that are not optimized with FOCUS



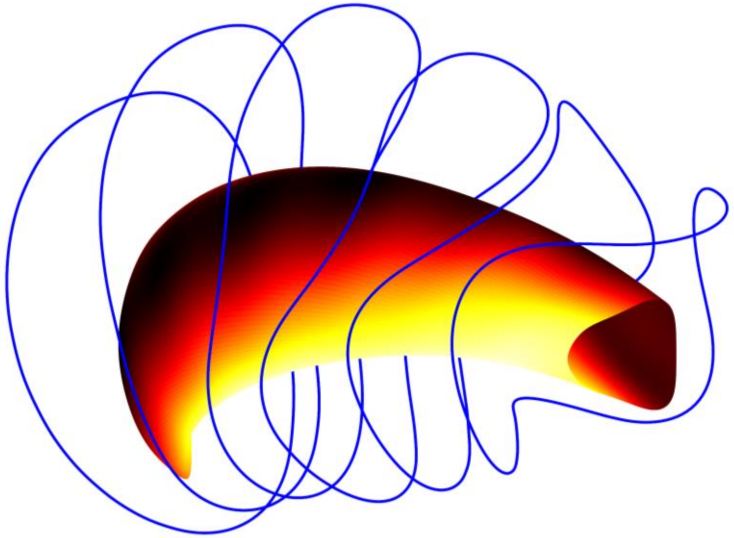
2. Coils optimized with no curvature functions and no coil to coil separation penalty function



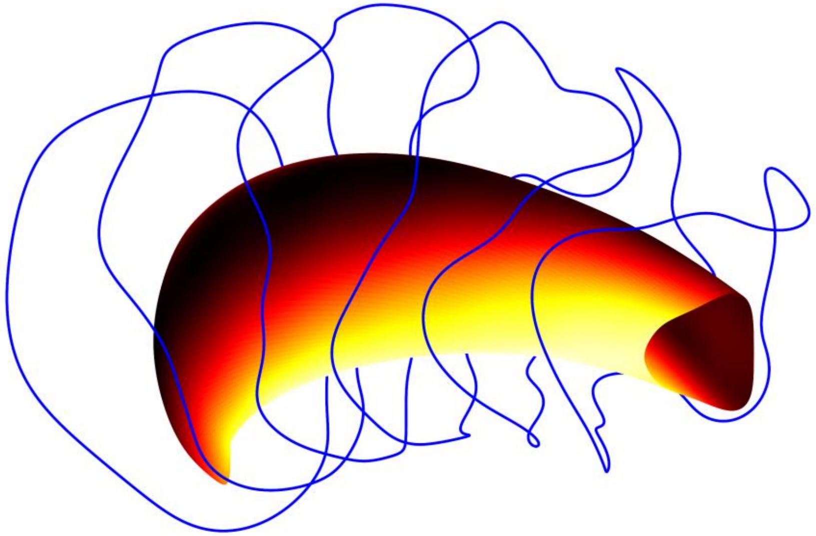
3. Coils optimized with the linear curvature objective function and no coil to coil separation penalty function



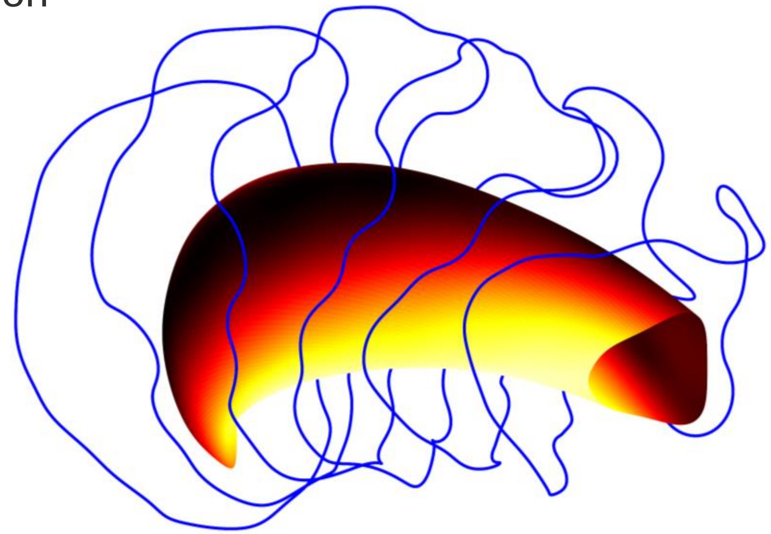
4. Coils optimized with the quadratic curvature objective function and no coil to coil separation penalty function



5. Coils optimized with a curvature penalty function and no coil to coil separation penalty function

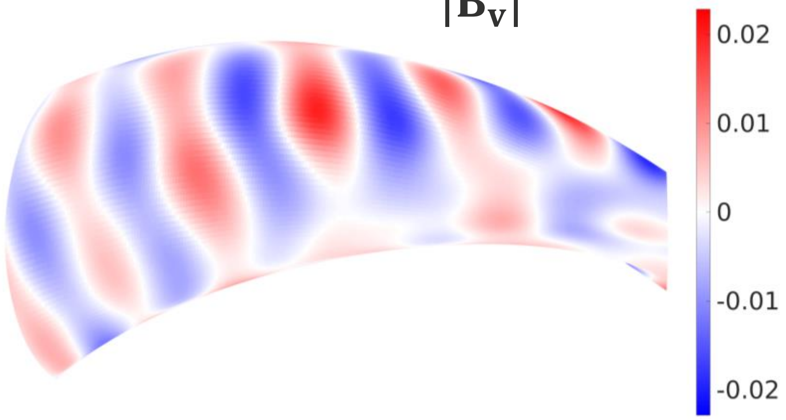


6. Coils optimized with a curvature penalty function and the coil to coil separation penalty function

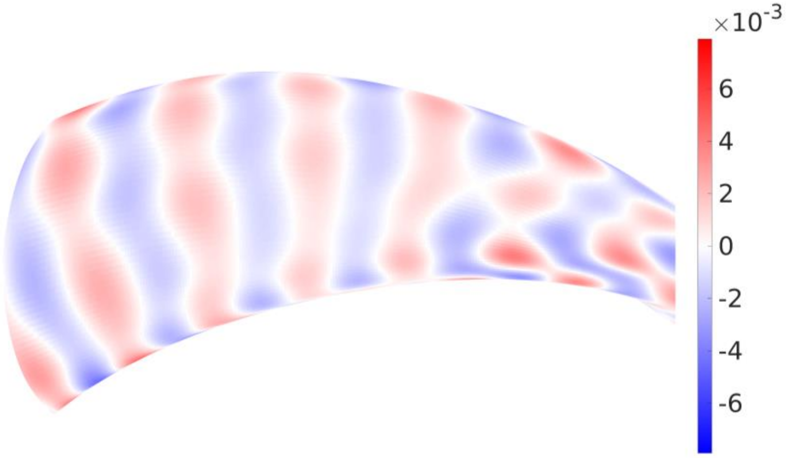


1. The HSX coils that are not optimized with FOCUS

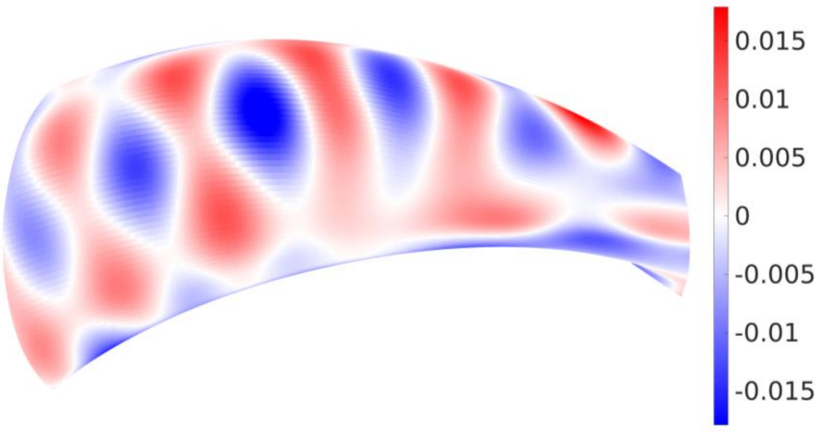
$$\frac{\mathbf{B_v} \cdot \mathbf{n}}{|\mathbf{B_v}|}$$



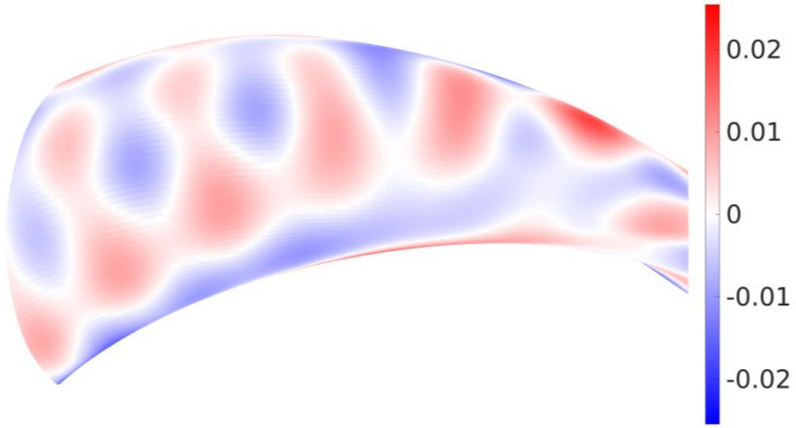
2. Coils optimized with no curvature functions and no coil to coil separation penalty function



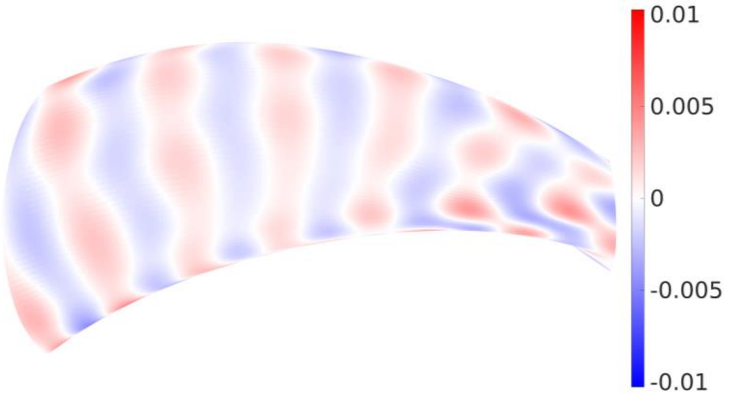
3. Coils optimized with the linear curvature objective function and no coil to coil separation penalty function



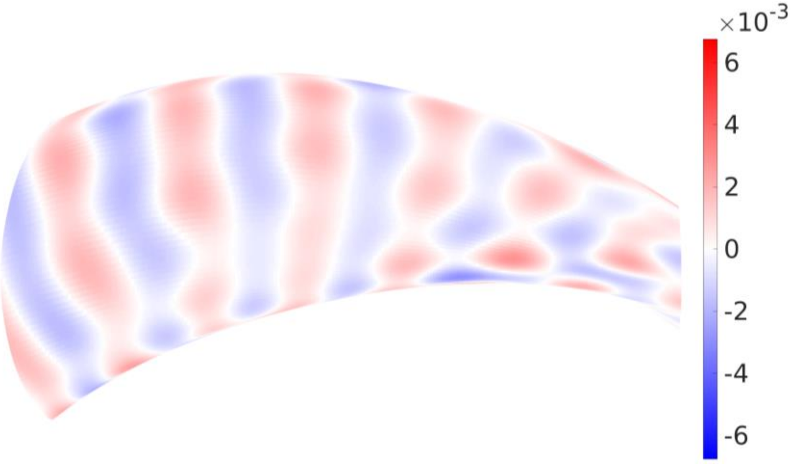
4. Coils optimized with the quadratic curvature objective function and no coil to coil separation penalty function



5. Coils optimized with a curvature penalty function and no coil to coil separation penalty function



6. Coils optimized with a curvature penalty function and the coil to coil separation penalty function



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	$\int_S \frac{1}{2}\left(\frac{\mathbf{B_v}\cdot\mathbf{n} - T_{Bn}}{\lvert\mathbf{B_v}\rvert}\right)^2 dS$	**Maximum Curvature (m^-1)**	**Minimum Coil to Coil Separation (cm)**
1	1.86x10^-5	12.3	9.3
2	8.40x10^-7	246	8.0
3	1.80x10^-5	20.0	8.5
4	1.90x10^-5	11.0	8.0
5	1.24x10^-6	12.3	8.5
6	8.50x10^-7	12.3	9.9

Conclusions and Future Work

- Penalty functions are used to constrain important engineering quantities in FOCUS
- Coils can be solved for that have both constrained engineering quantities and low magnetic field errors
- Penalty functions can be applied to various other problems, for instance constraining rotational transform profiles in STELLOPT to avoid low order rationales
- Coil to coil separation penalty function increases runtime and will be parallelized
- New coils for ATEN and BILA will be optimized to increase minimum coil to coil separation and hopefully decrease field error

Thanks for your attention